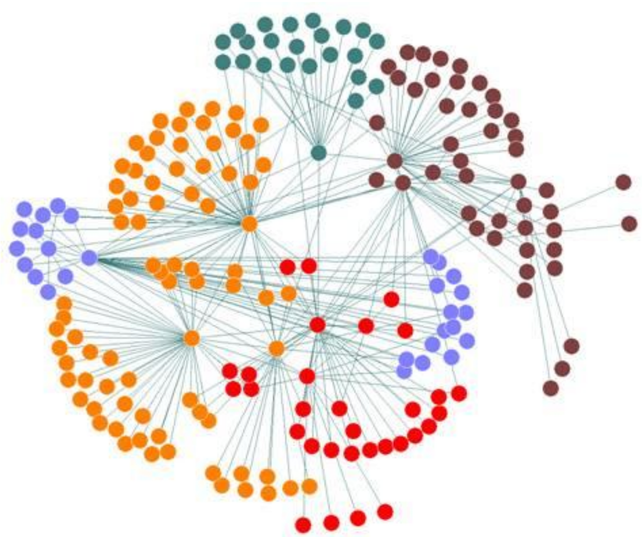
TEL AVIV UNIVERSITY

Algorithms and Applications in Social Networks



2023/2024, Semester A

Slava Novgorodov

Lesson #8

- Social learning
- DeGroot model
- Social Networks Riddles

Social Learning

Social Learning

- Social learning – process of changing opinion or behavior based on observation on others in the social circle

- No centralized mechanism for information aggregation

Examples of Social Learning

- Influential people (“celebs”) vs followers (following the opinion, “mimicking”)

- Group of good friends – each has some influence on others (different weights)

Examples of Social Learning

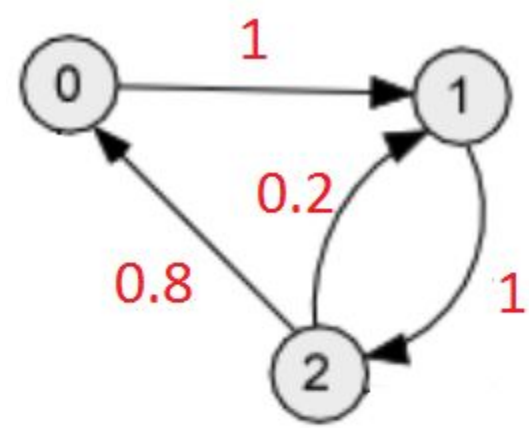
- Board of directors – need to get agreement on a decision.

Modeling Social Learning

Nodes: Directors
Links: Influence (“listens to”)
Weights: % of influence (sum up to 1)

Example:

- “0” listens to “1”
- “1” listens to “2”
- “2” listens to “0” (80%) and “1” (20%)



Modeling Social Learning

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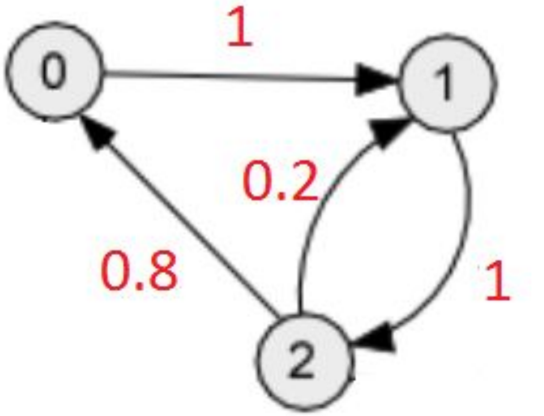


How to "guess" the final decision?

DeGroot Model

Reaching a consensus

"Reaching a consensus", by Morris DeGroot 1974

- Consensus - mutual agreement on a subject among group of people
  - Can a common belief be reached?
  - How long would it take?
  - How each individual belief contribute to consensus?
  - Which individuals have the most influence over final beliefs?

DeGroot Model

Nodes:

Graph with n nodes, each has an opinion:

$P_i(t) \in [0, 1]$ $i = \{1, \ldots, n\}$

Each node has an initial opinion $P_i(0)$

DeGroot Model

Opinion transition:

A matrix T, where T_{ij} is a weight on the opinion of other, i j - "how much i listens to opinion of j"

$$\sum_j T_{ij} = 1$$

Update

- per node: $$p_i(t+1) = \sum_j T_{ij} p_j(t)$$

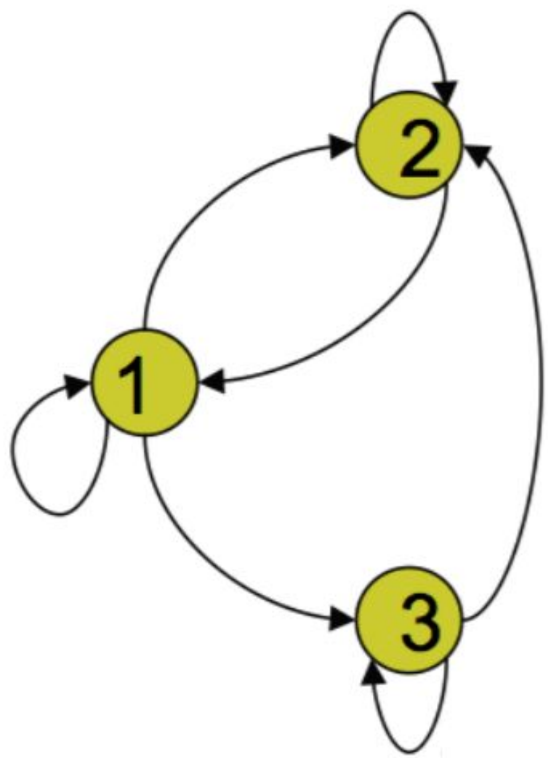
- vector of nodes: $$p(t) = Tp(t-1)$$

DeGroot Model

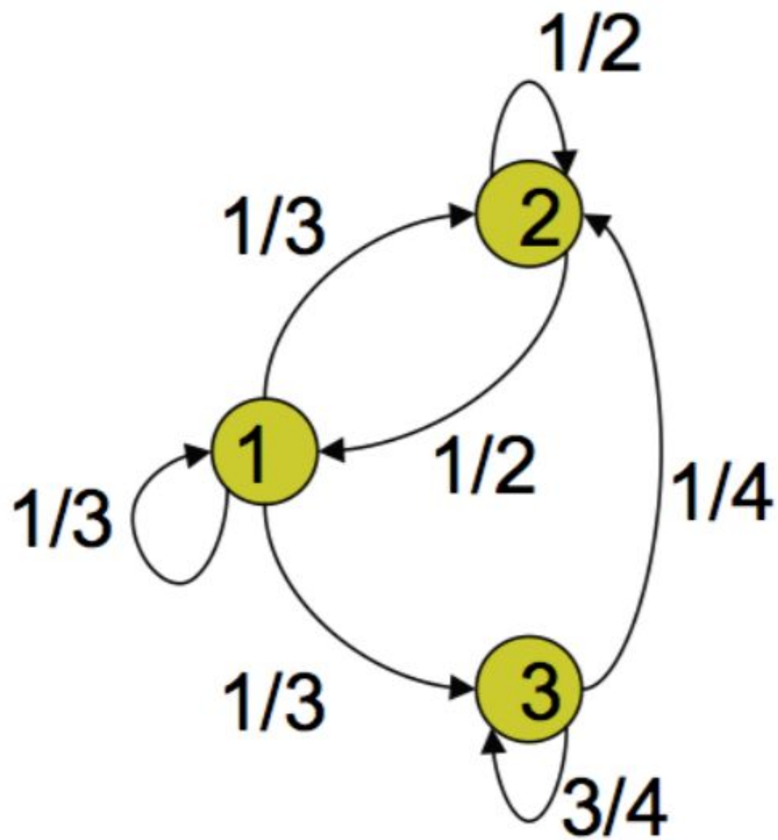
- Could a consensus be reached, i.e. all opinions converge to the same value?

$$\lim_{t \to \infty} p_i(t) = p^{\infty}$$

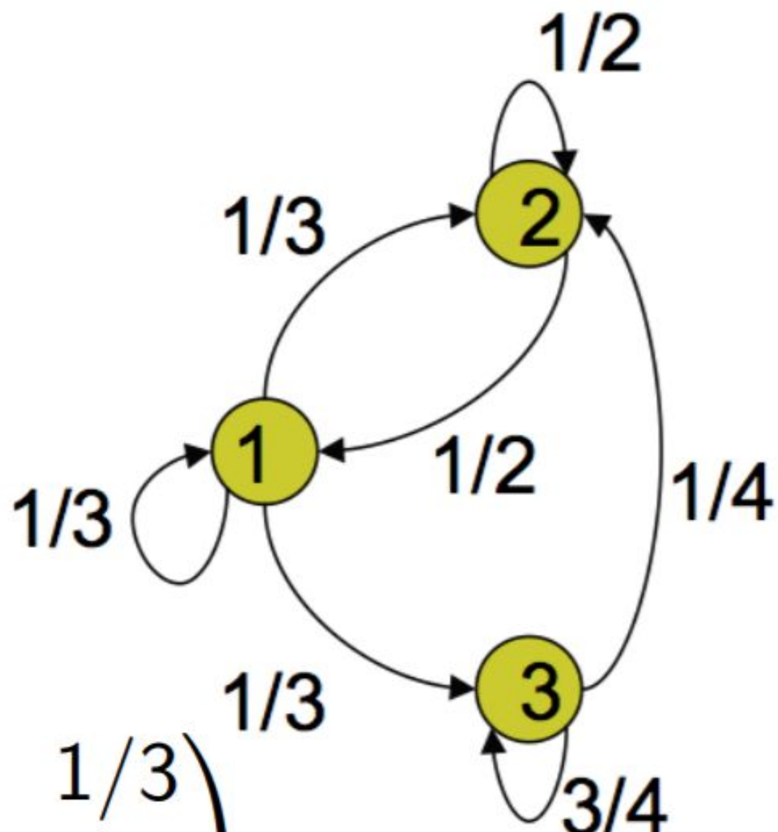
DeGroot Model – Example 1



DeGroot Model – Example 1



DeGroot Model – Example 1



$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

DeGroot Model – Example 1

$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

- Initial beliefs:

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Update step (1):

$$p(1) = Tp(0) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix}$$

DeGroot Model – Example 1

$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

- Previous step (1):

$$p(1) = \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix}$$

- Update step (2):

$$p(2) = Tp(1) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/18 \\ 5/12 \\ 1/8 \end{pmatrix}$$

DeGroot Model – Example 1

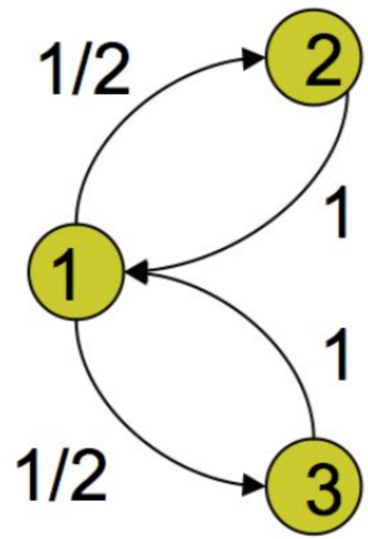
$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

- Consensus:

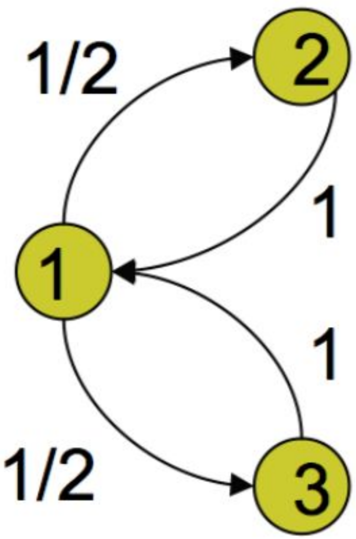
$$p(20) = Tp(19) = \begin{pmatrix} 3/11 \\ 3/11 \\ 3/11 \end{pmatrix}$$

$$p(21) = Tp(20) = p(20)$$

DeGroot Model – Example 2



DeGroot Model – Example 2



$$T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

DeGroot Model – Example 2

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$p(1) = Tp(0) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$p(2) = Tp(1) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Consensus?

DeGroot Model – Example 2

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$p(1) = Tp(0) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

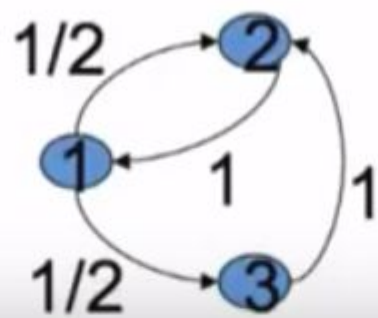
$$p(2) = Tp(1) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

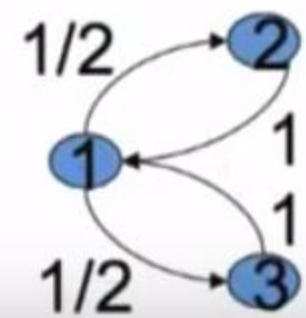
- Consensus? **No, the graph is periodic!**

Convergence

T *converges* if $\lim T^t b$ exists for all b

T is *aperiodic* if the greatest common divisor of its cycle lengths is one





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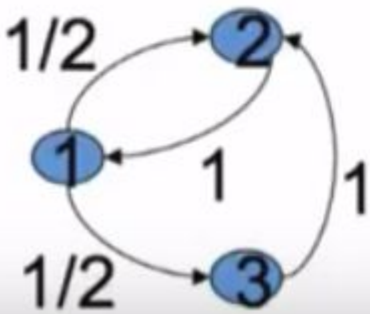
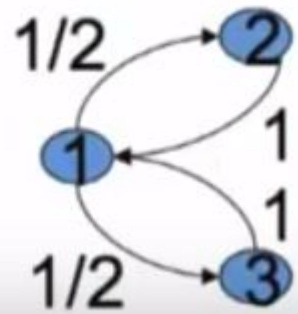


Aperiodic

gcd = 1

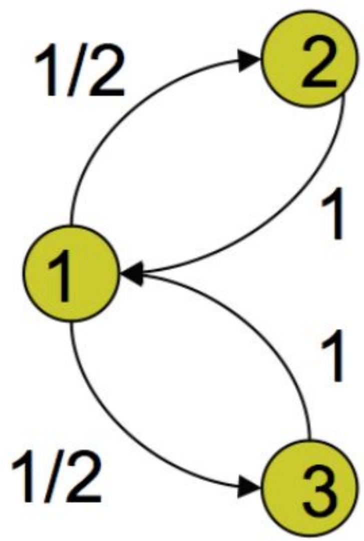


Periodic

gcd = 2

How to make graph aperiodic?

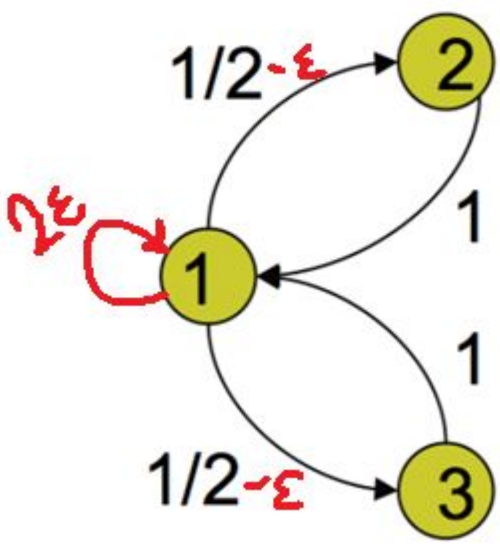
- Aperiodic iff gcd of all cycles length is 1!



How to make graph aperiodic?

- Aperiodic iff gcd of all cycles length is 1!

- Let's add self loop!



See recitation code for simulation!

DeGroot Model

- Observation:

$$p(t) = T \cdot p(t\text{-}1) = T \cdot T \cdot (p\text{-}2) = \ldots = T^t \cdot p(0)$$

Eventually:

$$p^\infty = \lim_{t \to \infty} p(t) = \lim_{t \to \infty} T^t p(0)$$

Perron – Frobenius Theorem

- For stochastic matrix (each row sums up to 1):

Let $\mathbf{T}$ *be a square 1) non-negative* $T_{ij} \geq 0$, *2) irreducible, 3) aperiodic stochastic matrix. Then there are exist*

$$\lim_{t \to \infty} T_{ij}^t = \pi_j$$

where

$$\pi_j = \sum_i \pi_i T_{ij}$$

$\pi = (\pi_1, \pi_2, ..\pi_n)$ *- is the left eigenvalue of* $\mathbf{T}$, *corresponding to* $\lambda_1 = 1$ *and* $\sum_i \pi_i = 1$

* Irreducible - G is strongly connected

Computing the belief

$$p(t) = Tp(t-1) = T^2 p(t-2) = T^t p(0)$$

$$p^\infty = \lim_{t \to \infty} p(t) = \lim_{t \to \infty} T^t p(0)$$

Computing the belief

$$p(t) = Tp(t-1) = T^2 p(t-2) = T^t p(0)$$

$$p^\infty = \lim_{t\to\infty} p(t) = \lim_{t\to\infty} T^t p(0)$$

$$\lim_{t\to\infty} T^t = \lim_{t\to\infty} \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \dots & \dots & \dots \\ T_{n1} & \dots. & T_{nn} \end{pmatrix}^t = \begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_n \\ \pi_1 & \pi_2 & \dots & \pi_n \\ \pi_1 & \pi_2 & \dots & \pi_n \end{pmatrix}$$

Computing the belief

$$p(t) = Tp(t-1) = T^2 p(t-2) = T^t p(0)$$

$$p^\infty = \lim_{t\to\infty} p(t) = \lim_{t\to\infty} T^t p(0)$$

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$$\mathbf{p}^\infty = \begin{pmatrix} \pi_1 & \pi_2 & \ldots & \pi_n \\ \pi_1 & \pi_2 & \ldots & \pi_n \\ \pi_1 & \pi_2 & \ldots & \pi_n \end{pmatrix} \begin{pmatrix} p_1(0) \\ \ldots \\ p_n(0) \end{pmatrix} = \begin{pmatrix} p^\infty \\ \ldots \\ p^\infty \end{pmatrix}$$

$$\pi \mathbf{T} = \pi \lambda$$

Back to Example 1

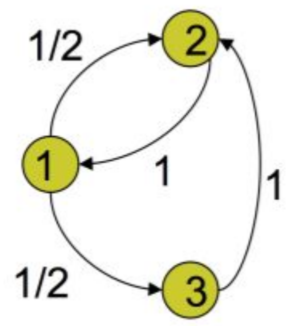
$$T^{20} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}^{20} = \begin{pmatrix} 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \end{pmatrix}$$

$$p^{\infty} = \begin{pmatrix} 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.27 \\ 0.27 \\ 0.27 \end{pmatrix}$$

$$\pi \mathbf{T} = \pi \lambda$$

$$\pi_1 = (0.27, 0.36, 0.36)$$

Equal Weight



$$p_i = \sum_j T_{ij} p_j = \sum_j \frac{A_{ij}}{d_i} p_j$$

$$p = (D^{-1}A)p$$

A – Adjacency matrix
d – degree

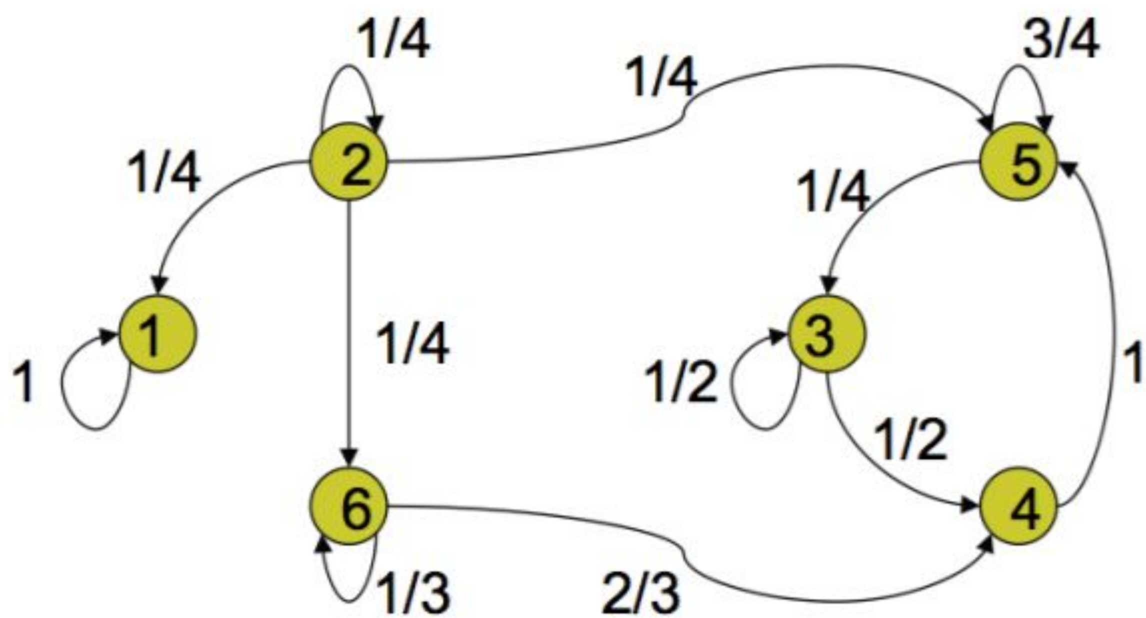
Closed sets

- A set of nodes C is called a closed set if there is no direct link from the node in C to the node outside C

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/4 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 2/3 \end{pmatrix}$$

Closed sets

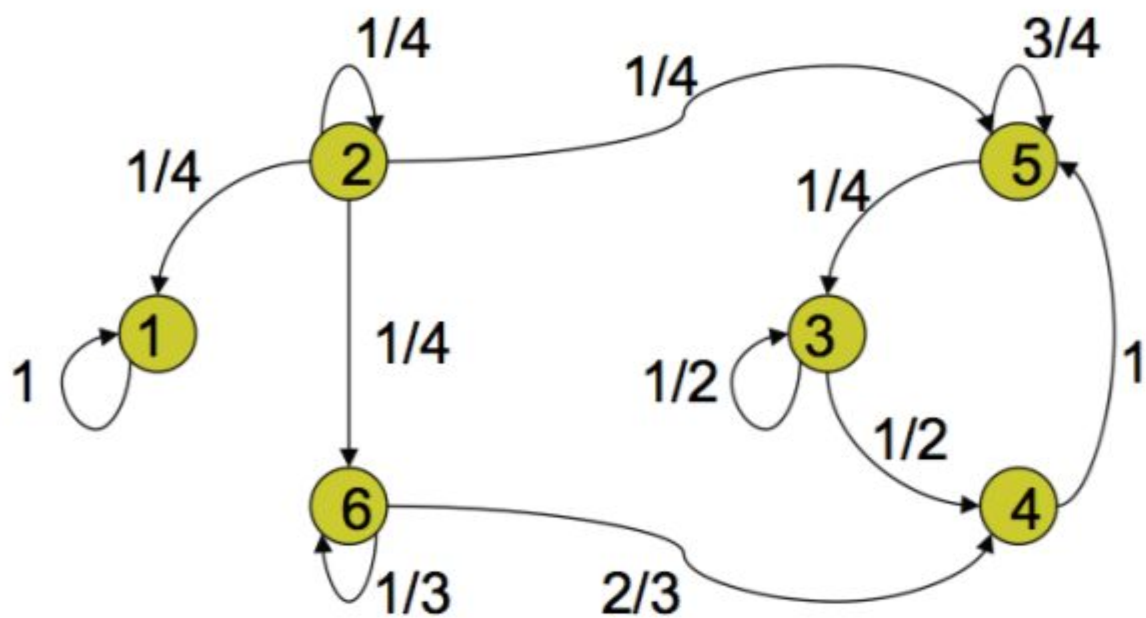
- A set of nodes C is called a closed set if there is no direct link from the node in C to the node outside C



Closed sets

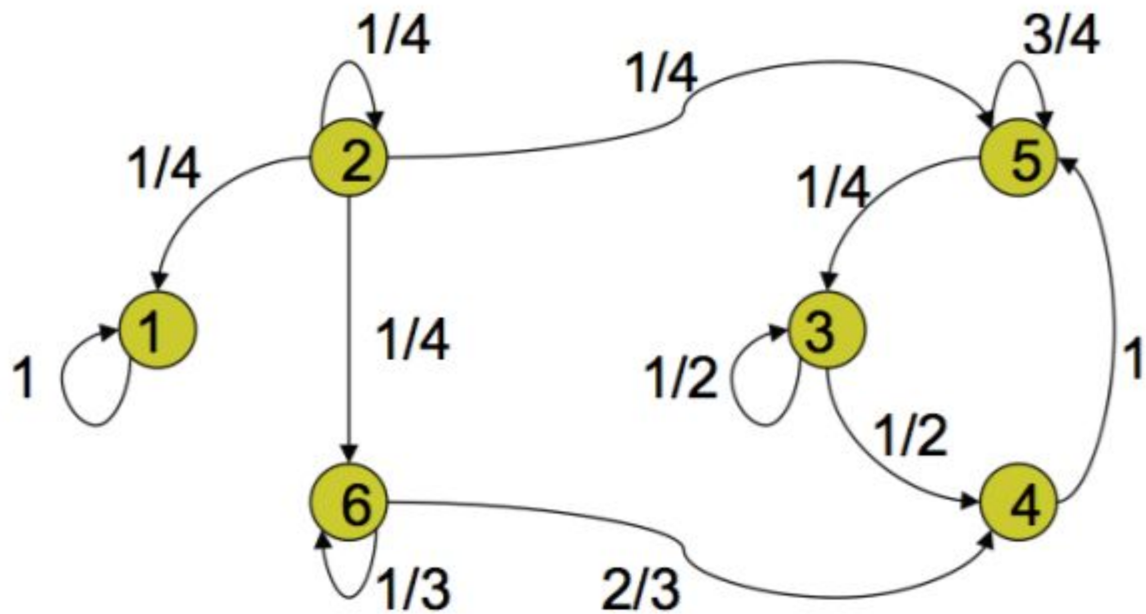
- Closed sets:

  {1}, {3,4,5}, {1,3,4,5}, {3,4,5,6}, {1,3,4,5,6}



Closed sets

- Which of them are also strongly connected?



Closed sets

- Which of them are also strongly connected?
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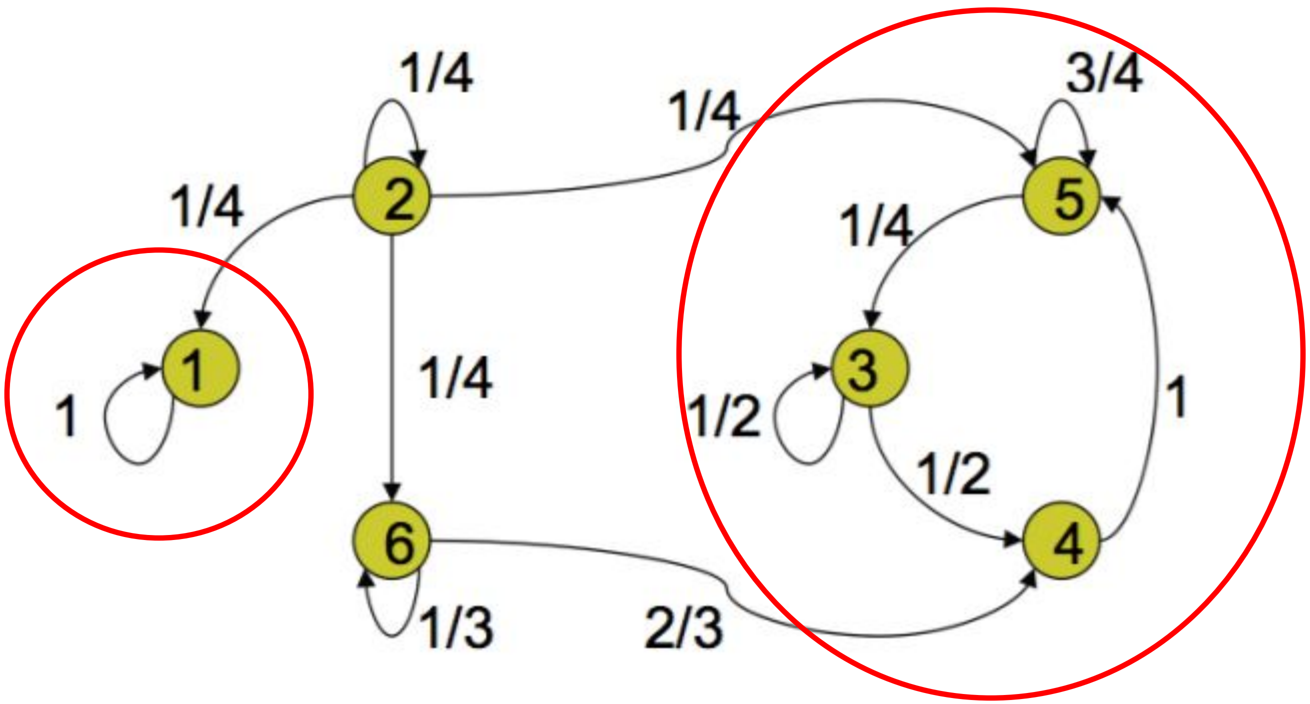


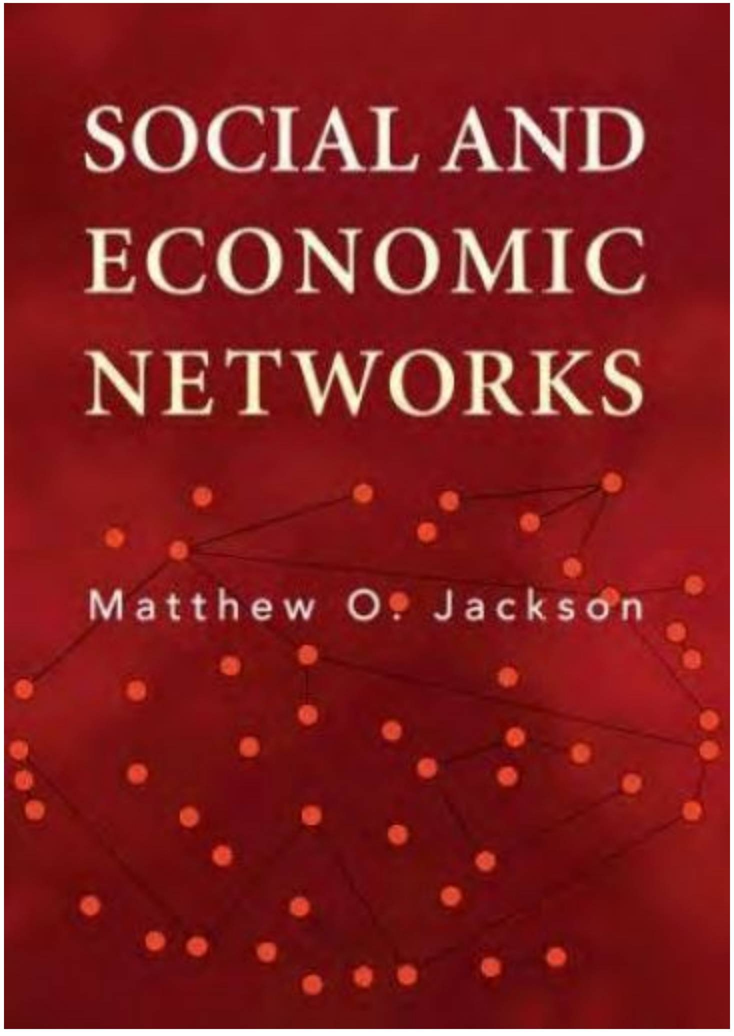
Closed sets

Theorem:

- T is convergent if and only if every set of nodes that is strongly connected and closed is aperiodic

- Every strongly connected closed and aperiodic set will reach own consensus.

Main reference



"Social and Economic Networks"
by Matthew O. Jackson
Chapter 8

Social Networks Riddles

Riddles (1)

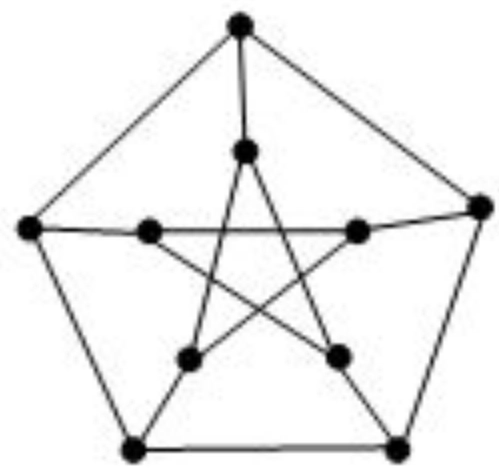
You need to plan a city metro network with n station. There are two “must-have” properties of this metro network:

1. Each station should be connected to 3 other stations.
2. Each station should be reachable within 2 “jumps”

What is the maximum number of n?

Riddles (1)

Solution: 10



Riddles (2)

1. A group of n people are connected each to other (clique), and using 2 ways of communications – phone and mail.

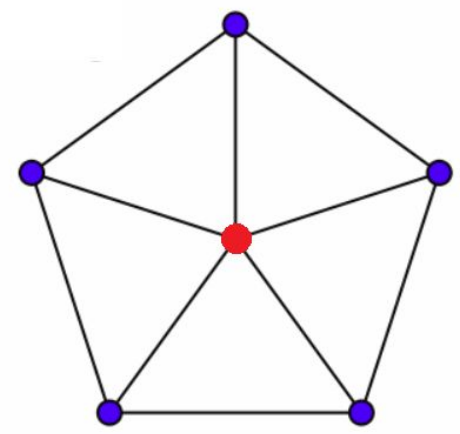
   Prove that they can decide to use only one of these two ways and still all of them will be reachable to each other (not necessarily directly connected)

Riddles (2)

Solution: Homework

Riddles (3)

n+1 people – CEO and n VPs.
Each VP connected to 2 other VPs and to CEO
in the following way (see the graph).



The connection is directed (one way)
Each person has at least one in and one out connection

Prove that every person can reach another!

Solution: in class…

Riddles (4)

Given a network of 9 nodes.

Each node’s degree is at least 4.

Prove that the graph is connected.

P.S. What is the general case?

Solution: in class…

Riddles

- Important connection between course material and real life problems
- Can be used also as interview questions
- Will probably appear in the exam ☺️

Thank you!

Questions?